数理解析研究所講究録
414巻 1981年 115-142

ON HILL'S EQUATION AND NEUMANN SYSTEMS

RIMS & UNIV. OF ARIZONA H. FLASCHKA

0. Preface (apology). This is not the talk given at the conference. Before the conference, I discovered (once again) errors in the main results, and so described only some fairly preliminary examples. I have not resolved the difficulties; hence, it might be better if I describe a new project, which is in any case more directly relevant to soliton research.

The question dealt with below is not too hard to explain. Let $q(x)$ be an N-soliton potential for Schrödinger's equation $-y''(x)+q(x)y(x)=Ey(x)$. If the N eigenvalues are $E=-\kappa_1^2,\dots,-\kappa_N^2$, and if $\phi_1,\dots,\phi_N$ are the corresponding eigenfunctions ($\int_{-\infty}^{\infty}\phi_j^2dx=1$), then

$$q(x)=-4\sum_1^N \kappa_j\phi_j^2(x).$$

This formula was found by Gardner, Green, Kruskal, Miura in CPAM (1974). Substitute it for $q(x)$ in

$$-\phi_j'' + q\,\phi_j = -k_j^2 \phi_j\,, \quad j=1,\dots,N\,,$$

to get

$$-\phi_j'' + \left(\sum_1^N -4k_i\phi_i^2\right)\phi_j = -k_j^2 \phi_j. \qquad (*)$$

This is a system of N nonlinear, coupled ordinary differential equations. One may expect (*) to have many interesting properties, since the N-soliton potentials of Schrödinger's equation do.

Is (*) completely integrable? (Yes. The solution was given in 1918 by R. Garnier).

Is there a Lax pair representation? (Yes. Implicit in Garnier's paper).

What does the <u>general</u> solution $\phi_1(x),\dots,\phi_N(x)$ of (*) have to do with Schrödinger's operator — for, <u>only</u> <u>one</u> solution will reproduce the N-soliton eigenfunctions. (D.V. and G.V. Choodnovsky, 1978: the general solution represents eigenfunctions of a quasiperiodic, N-gap potential — the "cnoidal" analog of the N-soliton potential).

PROBLEM. Find comparable results for other linear operators — $D^3 + qD + p$ (Boussinesq), Zakharov-Shabat, ..., for matrices (Toda lattice, ...).

This has two parts.

(i) Derive formulas for potentials in terms of eigenfunctions, analogous to $q = -4 \sum k_j \phi_j^2$.

(ii) Show that the system analogous to (*) is integrable.

At present, step (i) is fairly well understood, requiring some contour integrations quite familiar in inverse scattering calculations(+). When one deals with finite-gap potentials, there is a very beautiful method based on the associated Riemann surface theory. This was developed by I. Cherednik (Funkts. Anal. Prilozh., 1978), but not in enough detail to give all formulas in the needed form. Cherednik's method is illustrated below.

(+) D.J. Kaup, SIAM J. Appl. Math 1976 ; A.C. Newell, in Solitons, edited by R. Bullough; P. Deift and E. Trubowitz, CPAM 1979; P. Deift, F. Lund, E. Trubowitz, Comm. Math. Phys 1980. For Hill's equation, H.P. McKean and P. van Moerbecke, Invent. Math. 1975.

Step (ii), as far as I can tell, is not at all understood. In each case, integrability of the analog of (*) is shown by brute calculation: constants of motion are guessed (there <u>is</u> a common pattern), and vanishing of Poisson brackets is left to the reader. Below, I suggest a construction which will (if remaining details, and generalizations, go through) enable one to prove integrability for most other examples.

There is more to this than one might think at first glance.

a) There is a slightly different version of (*) (called C. Neumann's system), which J. Moser has shown to be intimately related to geodesic flow on an ellipsoid in $\mathbb{R}^{N+1}$ (notes of lecture at Chern Symposium, 1979). In turn, H. Knörrer (Invent. Math., 1980) has connected this geodesic flow with the algebraic geometry of quadrics. There is undoubtedly much interesting geometry to be discovered in the analogous problems for Boussinesq and other (non-hyperelliptic) soliton equations.

b) Analogs of (*) exist also for Painlevé equations (cf. a paper by Alan Newell and me, in preparation). A different representation, such as (*), may provide insights into this class of "well-behaved" systems.

c) A non-autonomous system, related in a certain way to (*), was encountered by Jimbo, Miwa, Mori, and Sato (Physica D, 1980) in a study of the impenetrable Boson gas. The relation of analogs of (*) to statistical mechanics models and holonomic quantum fields is still unexplored, but seems to be interesting enough to warrant study.

The first step, before one can hope to tackle the really interesting problems a), b), c), is to understand in a rational way what has so far been done mostly by experiment. I have tried to reproduce known results for Hill's equation, making sure at each stage that the generalization to Boussinesq was clear – at least in principle. Boussinesq, as well as other hyperelliptic problems, are at present being studied by R. Schilling (Univ. of Arizona), our results are still preliminary and not ready to be described here.

I next sketch the outlines of what I expect to become a general method, as it applies to Hill's equation. Most of the formulas required may be found in the survey by Date and Tanaka (Progr. Theor. Phys. Suppl. 59 (1976), 107-125), and in their recent monograph (I think so, because I could read the formulas, if not the 漢字 between them).

1. Trace Formulas.

I call a formula like $q = -4\sum_1^N k_j \phi_j^2$ a <u>trace formula</u>, following tradition, even though "traces" don't really come into play very much.

As an illustration of the origin of such formulas, I derive the trace formulas for the Jacobi matrix

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & & 0 \\ 0 & a_2 & b_3 & & 0 \\ & & & \ddots & \\ a_N & & & a_{N-1} & b_N \end{bmatrix}$$

familiar from Toda lattice theory (M. Toda, Physics Reports C, 1975). Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues, let $\phi^{(1)}, \dots, \phi^{(N)}$ be the normalized eigenvectors,

$$\phi^{(i)} \cdot \phi^{(i)} = 1, \qquad \phi^{(i)} \cdot \phi^{(j)} = 0 \quad \text{if } i \neq j ,$$

and let Φ be the matrix of eigenvectors (column i is $\phi^{(i)}$), $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$. Thus,

$$L\Phi = \Phi\Lambda ,$$

or

$$L = \Phi\Lambda\Phi^{-1} = \Phi\Lambda\Phi^{T} \qquad (1.1)$$

since Φ is an orthogonal matrix.

In (1.1), compare entries: it is very easy to see that (W. Ferguson, Math. Comp., 1980 uses these formulas for computing)

$$b_i = \sum_{k=1}^{N} \lambda_k \phi_i^{(k)2} ,$$

$$a_i = \sum_{k=1}^{N} \lambda_k \phi_i^{(k)} \phi_{i+1}^{(k)}$$

(subscript denotes component; by convention, $\phi_{N+1}^{(k)} := \phi_1^{(k)}$).
This is the analog of $q = -4 \sum k_j \phi_j^2$.

If one substitutes these formulas into $L\phi^{(j)} = \lambda_j \phi^{(j)}$, or into the difference equation $a_{n-1}\phi_{n-1}^{(j)} + a_n \phi_{n+1}^{(j)} + b_n \phi_n^{(j)} = \lambda_j \phi_n^{(j)}$, one gets the difference analog of (*) (which has not, so far, been studied). If one lets the matrix L change according to $\dot{L} = [B, L]$,

$$B = \begin{bmatrix} 0 & a_1 & 0 & \dots & a_N \\ -a_1 & 0 & a_2 & & \\ 0 & -a_2 & 0 & & \\ -a_N & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

(basically the Toda equations), so that $\dot{\phi}^{(j)} = B\phi^{(j)}$, or $\dot{\phi}_n^{(j)} = a_n\phi_{n+1} - a_{n-1}\phi_{n-1}$, and if one then replaces a_n by $\sum \lambda_k \phi_n^{(k)}\phi_{n+1}^{(k)}$, one gets a system of coupled nonlinear differential equations for $\phi_n^{(j)}$ which contains the Toda equations. The system is reduced to fewer unknowns in Deift, Lund, Trubowitz, loc. cit.

This idea of rewriting linear eigenvalue problems, and their isospectral flows, in a "self-consistent" way by coupling together all the important eigenfunctions is very powerful; it was introduced by Deift and Trubowitz (CPAM, 1978) in a thorough study of inverse scattering for $-D^2+q$, and is used in their forthcoming analysis of scattering theory (or rather, inverse spectral theory) for D^3+qD+p.

Unfortunately, no other trace formulas seem to be as simple as those for the Jacobi matrix. In general, one needs detailed information about how the eigenfunctions depend on the spectral parameter. Then one applies Cauchy's theorem to a cleverly chosen integrand: $\frac{1}{2\pi i}\oint_{|z|=R} \cdots\, dz = 0$ turns, for example, into $q - 4\sum k_j \phi_j^2 = 0$.

In a moment, this idea will become completely obscure thanks to the introduction of abelian differentials, non-special divisors, and other recondite concepts. To argue that the procedure is, in any concrete case, just contour integration (possibly, with complicated branch cuts), I show how to derive the by now popular formula $q = -4\sum k_j \phi_j^2$.

The Jost functions for the N-soliton potential are conveniently derived by Bäcklund transformations; from, e.g., the formulas given in Flaschka-McLaughlin, Springer Lecture Notes 515, one may derive

$$f_1(x,k) = \frac{1}{\prod_1^N (k_j - ik)} \frac{W(e^{ikx}, \cosh\theta_1, \ldots, \cosh\theta_N)}{W(\cosh\theta_1, \ldots, \cosh\theta_N)},$$

where W denotes the Wronskian, $\theta_i = k_i(x - x_i)$, x_i a constant. f_1 is the solution of $-y'' + qy = k^2 y$ which goes like e^{ikx} as $x \to \infty$. Since $e^{k_j x} + c_j e^{-k_j x}$ is proportional to $\cosh k_j(x - x_j)$ for some suitable c_j, one finds that $f_1(x;ik_j) = c_j f_1(x,-ik_j)$ for a certain c_j. (This is the basis of Date's direct soliton construction, see: Proc. Japan Acad., 55 (1979), 27).

One now integrates

$$k f_1(x,k)\, f_1(x,-k)\, dk$$

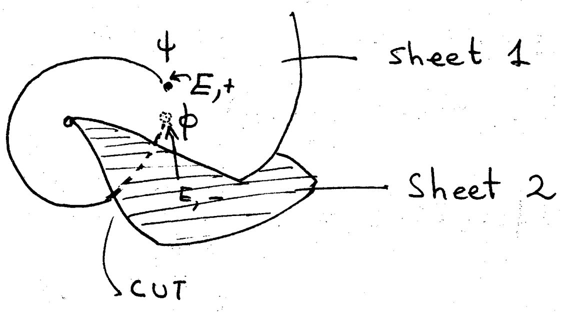
around a circle $|k| = R$ big enough to enclose all ik_j. At $k=\infty$, the residue is $-\frac{1}{2}q^{(\dagger)}$, and at the finite poles, $\pm ik_j$, the residue is proportional to $f_1(x,ik_j)^2 = -\beta_j \phi_j^2(x)$. The normalization constant, needed to make $\int_{-\infty}^{\infty}\phi_j^2 = 1$, gives exactly the required formula.

Now to Hill's equation. If q is a finite-gap potential, the analog of the Jost function f_1 is the Baker-Ahiezer function $\psi = \psi(x,P)$. It is meromorphic for P on a Riemann surface $\mathcal{R}$: $R^2 = \prod_0^{2N}(E - E_i^\circ)$, has poles at points $D_1, \dots, D_N \in \mathcal{R}$ (independent of x), and goes like $e^{kx}(1 + O(k^{-1}))$ at $E=\infty$, $k = \sqrt{E}$. The analog of $f_1(x,-k)$ is a function $\phi = \phi(x,P)$ which, in this simple case, is just ψ on the other sheet. More precisely:

(†) Any product F of two solutions of $-y'' + qy = k^2 y$, such as $F = f_1(x,k) f_1(x,-k)$, solves $\frac{1}{4}F''' - qF' - \frac{1}{2}q'F = -EF'$. Expand $F = 1 + cE^{-1} + \cdots$ at $E=\infty$.

The surface $\mathcal{R}$ is the 2-sheeted cover of the complex E-plane, cut so as to make $\sqrt{\prod_0^{2N}(E-E_i^0)}$ singlevalued. A point $P \in \mathcal{R}$ is $E \in \mathbb{C}$ together with a $\pm$ sign to determine the sheet; if $P = (E, \pm)$, let $\tau P = (E, \mp)$ be the point on the other sheet. Then

$\phi(x, P) = \psi(x, \tau P)$, the continuation of ψ around a branch point.



In the soliton case, $f_1(x,k)\,f_2(x,-k)$ automatically has poles at the desired points $\pm ik_j$, and only there. In the finite-gap case, ψ has poles at D, $D = \{D_1, \dots, D_N\}$, ϕ at $\tau D = \{\tau D_1, \dots, \tau D_N\}$. Before integrating $\phi\psi$ around a big circle, one must take out the extra poles $D + \tau D$, and put in the "correct" ones at the "eigenvalues". D and τD come from having genus > 0.

Denote the points D_i, τD_i on $\mathcal{R}$ by $(\mu_i, \pm)$, $(\mu_i, \mp)$, respectively (i.e., they lie on opposite sheets). From

Date-Tanaka, we have

$$\psi(x,P)\phi(x,P) = \prod_1^N \frac{(E-\mu_j(x))}{(E-\mathring{\mu}_j)}$$

for certain quantities $\mu_j(x)$ ("tied spectrum").

Let

$$\omega = \frac{1}{2}\frac{\prod_1^N (E-\mathring{\mu}_j)}{\sqrt{\prod_0^{2N}(E-E_j^\circ)}}\, dE.$$

ω has no finite poles(+), and goes like $-\frac{1}{E}\, dE^{-1/2}$ at $E=\infty$ (a branch point).

$$\phi(x,P)\psi(x,P)\,\omega = \frac{1}{2}\frac{\prod_1^N (E-\mu_j(x))}{\sqrt{\prod_0^{2N}(E-E_j)}}\, dE$$

has no finite poles, and so cannot have a residue at ∞: an integral around a big circle would vanish (as would the integral of $f_1(x,k)f_1(x,-k)$, without the extra factor of k).

One now multiplies $\phi\psi\omega$ by something analogous to the k that multiplied $f_1(x,k)f_1(x,-k)$: a function which has poles at the "eigenvalues", the analogs of $\pm ik_j$,

---

(+) i.e., $\frac{dE}{\sqrt{E-E_j^\circ}}$ is integrable near E_j°.

and produces a pole with residue at ∞.

In previous papers (e.g., Moser's), the "eigenvalues" have been chosen at $E_0^\circ, E_2^\circ, E_4^\circ, \dots$ (the left endpoints of the branch cuts), but there is no compelling reason, in the finite-gap case, to put the "eigenvalues" there; the generalization of the $\kappa_1, \dots, \kappa_N$ of the N-soliton potential is really the whole Riemann surface $\mathcal{R}$(†). Indeed, I get some new results by placing the "eigenvalues" arbitrarily. Let $E_0, \dots, E_N$ (no superscript "0" this time) be distinct finite numbers, and let $P_0, \dots, P_N$ be a choice of points on $\mathcal{R}$ where $E = E_j$, i.e., $P_j = (E_j, +)$ or $(E_j, -)$. There exists a unique function h on $\mathcal{R}$ which has simple poles at the P_j, and a simple zero at $E = \infty$, $h \sim E^{-1/2}$ at ∞ (††).

(†) When one considers ∞-gap potentials, as do McKean and Trubowitz, the choice of $E_0, \dots$ may well matter. This I don't know.

(††) $h(P) = \dfrac{R(P) + p(E(P))}{\prod_1^N (E(P) - E_j)}$, where: $R = \sqrt{\prod_0^{2N} (E - E_j^\circ)}$,

p is a polynomial of degree $\leq N$ for which $p(E_i) = R(P_i)$. The denominator is zero at $E = E_j$ on both sheets; the numerator is $R(P_j) + R(P_j)$ on one sheet, $-R(P_j) + R(P_j)$ on the other.

Finally, consider the differential $h\phi\psi\omega$. It has poles at $P_0, \ldots, P_N$, with residues $\beta_j \phi(x, P_j)\psi(x, P_j) := \beta_j \phi_j \psi_j$, and a pole at ∞ with residue -1 (one sees this by expanding the various functions). The sum of residues $= 0$, and so

$$1 = \sum_0^N \beta_j \phi_j \psi_j . \tag{1.2}$$

Now multiply by E: as for the residues of $Eh\phi\psi\omega$, nothing changes except a factor E_j is introduced at P_j, while the residue at ∞ becomes $\frac{1}{2}q + h_3$, where h_3 is the coefficient in the expansion $h = E^{-1/2} + h_2 E^{-1} + h_3 E^{-3/2} + \ldots$ at $E = \infty$. Hence

$$q = 2h_3 - 2\sum_0^N E_j \beta_j \phi_j \psi_j . \tag{1.3}$$

This is the periodic analog of our original $(*)$. The remarkable thing is that Cherednik's method removes all vestiges of guessing from the various steps. Here is a completely general version of what I have described.

Let $\mathcal{R}$ be a Riemann surface, with distinguished point ∞. Let $E : \mathcal{R} \to \mathbb{C} \cup \{\infty\}$ be meromorphic on $\mathcal{R}$, with a single, n^{th} order, pole at ∞. Let D be a nonspecial divisor of degree $= g =$ genus of $\mathcal{R}$ ($\mathcal{R}$ is, of course, compact and smooth). There exists a unique function $\psi(x,P)$, $P \in \mathcal{R}$, which is meromorphic on $\mathcal{R}-\{\infty\}$, has poles in D, and goes like $e^{kx}\{1+O(k^{-1})\}$ at ∞ ($k = E^{1/n}$). By Riemann-Roch, there further exist: a nonspecial divisor τD, and a differential ω which has as its only pole the point ∞, where it behaves as $-k^2\,dk^{-1}$, and which has zeros at $D+\tau D$. There exists (by the same argument that produces ψ; see McKean, Springer Lecture Notes 755) a unique $\phi(x,P)$ with pole divisor τD, going like $e^{-kx}\{1+O(k^{-1})\}$ at ∞.

This is a simple and beautiful step. Finite-gap theory starts with the Baker-Ahiezer function ψ. Now, ϕ and ω, introduced very explicitly for Hill's equation earlier, arise completely naturally, their existence and uniqueness being guaranteed by general Riemann surface theory.

Now let h be a function that has simple poles at $P_0, \dots, P_N \in \mathcal{R}-\{\infty\}$ (in general, $N \neq g$). The statement "sum of residues of $E^{\ell} h \psi \phi \omega = 0$" will produce trace formulas like (1.2),(1.3). Cherednik develops a calculus that makes it possible to compute residues at ∞ of $\phi\psi\omega$ from the coefficients of the operator L_n (analog of $-D^2+q$) that has ψ as eigenfunction, $L_n\psi = E\psi$.

Everything is very intrinsic to $\mathcal{R}$, but when one leaves the hyperelliptic case, not much can be written down explicitly. Even for Boussinesq (operator D^3+qD+p), I don't know what ω and h look like, and there are independent trace formulas gotten from $\sum_{\text{Res}} E^\ell h\psi'\phi\omega = 0, \dots$. These are matters currently being investigated by Schilling and me. Adaptation of these ideas to 2×2 Zakharov-Shabat systems and 2^{nd}-order difference equations is simpler; it will be described in a paper in preparation.

2. Neumann and Rosochatius Systems.

It is now tempting to substitute (1.3) into $-\psi_i'' + q\psi_i = E_i\psi_i$, $-\phi_i'' + q\phi_i = E_i\phi_i$, to produce

$$\begin{cases} -\psi_i'' + \left(2h_3 - 2\sum E_j\beta_j\phi_j\psi_j\right)\psi_i = E_i\psi_i \\ -\phi_i'' + \left(2h_3 - 2\sum E_j\beta_j\phi_j\psi_j\right)\phi_i = E_i\phi_i \,. \end{cases} \tag{2.1}$$

Garnier solved (2.1) in 1918, as byproduct of a study of reduced Schlesinger equations; D.V. and G.V. Choodnovsky developed the theory of (2.1) recently, in connection with Hill's equation. As far as I can tell, (2.1) has a severe defect. Even though it is an integrable system, with Lax pair, its Riemann surface has genus $N+1$, higher by 1 than the genus of the Riemann surface of Hill's equation, with which everything started. My aim is to construct the Lax pair for (2.1) from the original data — $\mathcal{R}$, D, ψ, ϕ, ω, h —, and I do not see how to increase the genus of my Riemann surface in any sensible way.

Hence, I now describe a subtle and very interesting modification of (2.1), which leads to Hamiltonian systems studied by Carl Neumann in 1859, and by E. Rosochatius in 1877 (quoted by Moser).

With $'$ denoting $\frac{d}{dx}$, $(h\phi\psi\omega)''$ is again an abelian differential, and Σ Residue $=0$ says

$$q\sum \beta_j\phi_j\psi_j - \sum E_j\beta_j\phi_j\psi_j + \sum \beta_j\phi_j'\psi_j' = 0 ;$$

since, by (1.2), $\sum \beta_j\phi_j\psi_j = 1$, this gives

$$q = \sum \beta_j E_j \phi_j \psi_j - \sum \beta_j \phi_j' \psi_j . \tag{2.2}$$

Instead of (2.1), there arises the system

$$\begin{cases} -\psi_i'' + \left(\sum \beta_j E_j \phi_j \psi_j - \sum \beta_j \phi_j' \psi_j\right) \psi_i = E_i \psi_i , \\ -\phi_i'' + \left(\sum \beta_j E_j \phi_j \psi_j - \sum \beta_j \phi_j' \psi_j\right) \phi_i = E_i \phi_i . \end{cases} \tag{2.3}$$

This is to be supplemented by (1.2),

$$\sum \beta_j \phi_j \psi_j = 1 , \tag{2.4}$$

and the x-derivative of (2.4),

$$\sum \beta_j (\phi_j' \psi_j + \phi_j \psi_j') = 0 . \tag{2.5}$$

One can check that if (2.4), (2.5) hold initially, they are preserved by (2.3). Thus, (2.3 -5) is a system with $2(N+1) - 2 = 2N$ degrees of freedom. This system, furthermore, is invariant under scaling $\psi_i \to \alpha \psi_i$, $\phi_i \to \alpha^{-1} \phi_i$; this removes N more degrees of freedom. If one sets

$$\sqrt{\beta_j}\, \psi_j = r_j e^{\theta_j} , \quad \sqrt{\beta_j}\, \phi_j = r_j e^{-\theta_j} ,$$

$$\sqrt{\beta_j}\, \psi_j' = s_j e^{\theta_j} + \xi_j r_j e^{\theta_j} , \quad \sqrt{\beta_j}\, \phi_j' = s_j e^{-\theta_j} - \xi_j r_j e^{-\theta_j} ,$$

then $2\xi_j r_j^2 := \gamma_j = \text{const.}$, and after a reduction very like the elimination of angular momentum in the Kepler problem, one finds the following system for r_j and s_j :

$$\ddot{r}_j = -E_j r_j - \frac{\gamma_j^2}{r_j^3} + \left(\sum E_i r_i^2 + \sum \frac{\gamma_i^2}{r_i^2} - \sum \dot{r}_i^2 \right) r_j . \qquad (2.6)$$

This system was studied (and, apparently, solved) by Rosochatius. Notice that the reduction by scaling is inapplicable if E_j is a branch point of R, for then $\phi_j = \psi_j$. If <u>all</u> E_j are branch points, the two pairs of systems (2.3) become identical, since $\phi_j = \psi_j$ for each j, and we get the <u>Neumann system</u>

$$-\psi_i'' + \left(\sum E_j \beta_j \psi_j^2 - \sum \beta_j \psi_j'^2 \right) \psi_i = E_i \psi_i ,$$

subject still to (2.4), (2.5).

Moser (Chern lecture, loc. cit.) attributes the reduction of (2.3) to (2.6) to P. Deift, but it was not realized that Rosochatius' system arises naturally from the general trace formula for Hill's equation.

3. A few remarks about Neumann and Rosochatius Systems.

This is where <u>really</u> interesting things finally begin to happen, but to describe them would just be a repetition of the papers cited already. A few remarks, to indicate why anyone might want to study these funny systems further, may nonetheless be in order.

From now on, I look only at Neumann's system; setting $x_i = \sqrt{\beta_i}\,\psi_i$, $y_i = \sqrt{\beta_i}\,\psi_i'$, I write it in the usual form

$$x_i'' = -E_i x_i + \left(\sum E_j x_j^2 - \sum y_j^2\right) x_i \qquad (2.7)$$

subject to

$$\sum x_j^2 = 1, \quad \sum x_j y_j = 0. \qquad (2.8)$$

(i) Neumann's system may be interpreted as the equations describing free oscillators, $x_i'' = -E_i x_i$, constrained to the sphere $\sum x_j^2 = 1$ in configuration space. This interpretation is shown to have analogs for the linear problems of Sine-Gordon, NLS, and the Toda lattice, by Deift, Lund, Trubowitz (loc. cit.).

There is no a-priori reason for the free system $x_i'' = -E_i x_i$ to produce an integrable system upon constraint to the intersection of quadrics (2.8). It would be very interesting to have this mechanism explained.

ii) The Hamiltonian system with Hamiltonian

$$H = \frac{1}{2}\sum E_i x_i^2 + \frac{1}{2}\left(\sum x_i^2 \sum y_i^2 - \left(\sum x_i y_i\right)^2\right)$$

is integrable; N+1 integrals in involution are

$$F_i = x_i^2 + \sum_{j(\neq i)} \frac{(x_i y_j - x_j y_i)^2}{E_i - E_j} .$$

iii) If this Hamiltonian system is reduced by elimination of the integral $\sum x_i^2$ and its conjugate variable, the Neumann system is obtained. Hence, the latter is integrable, as it inherits the constants of motion of the system in ii).

iv) The Hamiltonian flow of ii) preserves the spectrum of

$$\tilde{L}(x,y) = (I - P_x)(\mathcal{E} - y \otimes y)(I - P_x);$$

here $(y \otimes y)_{ij} = y_i y_j$, $\mathcal{E} = \mathrm{diag}(E_0, \ldots, E_N)$, and P_x is projection onto the vector x,

$$P_x = \frac{x \otimes x}{|x|^2}, \quad P_{x\,ij} = \frac{x_i x_j}{|x|^2} .$$

v) (This statement is deliberately incorrect).

Take a line through $x = (x_0 \ldots x_{N+1}) \in \mathbb{R}^{N+1}$ in the direction $y = (y_0 \ldots y_{N+1})$. Under the Hamiltonian flow ii), the point of tangency of this line with the ellipsoid $\sum \frac{x_i^2}{E_i} = 1$ moves along a geodesic. The eigenvalues of $\tilde{L}(y,x)$ (which are NOT E_j) are those numbers λ_j for which the line $x+sy$ touches the quadric $\sum \frac{x_i^2}{\lambda_i - E_i} + 1 = 0$, confocal with $\sum \frac{x_i^2}{E_i} = 1$. At the point of contact, the normal to the quadric is the eigenvector of L for eigenvalue λ_j.

What is incorrect is that the flow for which all these miraculous things happen is not the flow (ii) I have written, but it is close enough. As Moser explains in detail, this geometric interpretation provides the first example in which every aspect of an $\dot{\tilde{L}} = [\tilde{B}, \tilde{L}]$ isospectral deformation admits a concrete translation (the higher constants of motion excepted).

One quite naturally wants to generalize this: what is the next most complicated geometrical setting? Presumably, it is associated with the Boussinesq 3rd-order operator, $D^3 + qD + p$. Here is the problem: <u>none of the results i) - v) just quoted were derived with any reference to Hill's equation.</u> At best, the matrix $\tilde{L}$ emerges as one of a huge class of matrices having analyzable isospectral deformations by virtue of lying on a Kac-Moody lie algebra orbit (Adler - van Moerbecke, Adv. Math. 1981). This matter is taken up next.

5. On L-B pairs.

Adler- van Moerbecke (loc.cit.) show that the Neumann equations imply a commutator equation of the form (one should set the coefficient of each power of h equal to 0 separately)

$$\dot{\tilde{B}}(h) = [\tilde{B}(h), \tilde{L}(h)], \qquad (*)$$

where $\tilde{B}(h) = \mathcal{E}h^2 + h(x\otimes y - y\otimes x) - x\otimes x$,

$\tilde{L}(h) = \mathcal{E}h + x\otimes y - y\otimes x$, $\mathcal{E} = \mathrm{diag}(E_0, \ldots, E_N)$. $\dot{}$ means differentiation with respect to the x of Hill's equation; to avoid confusion with the vector $x = (x_0, \ldots, x_N)$, I take $\dot{} = \frac{d}{dt}$. Polynomials in a parameter such as h, with matrix coefficients belonging to some Lie algebra $\mathfrak{g}$, are said to belong to a "Kac - Moody extension" of $\mathfrak{g}$. Adler - van Moerbecke arrive at (*) as a very special (low-dimensional) case of spinning-top type equations on Lie algebras. By expanding (*) about $h = 0$, one arrives at Moser's $\tilde{L}(x,y)$.

In principle, one can search for more complicated (higher degree of freedom) versions of (*) in a systematic way, by following Adler-van Moerbecke's prescriptions. To get, for example, the corresponding Kac-Moody systems for the Boussinesq or DNLS linear problem, one must convert the latter into Neumann type equations and compare with candidates for (*). That is, one works from both ends and hopes to meet in the middle.

It would be much more efficient to derive (*) directly from the data given about Hill's equation, and this is the project.

First of all, it should be noted that every Kac-Moody equation of the type (*) is equivalent to a commutator condition on differential (or integro-differential!) operators. Suppose L, B are matrix differential operators, and $[L,B]=0$. Then let $C(x,x_0,E)$ be a solution matrix of $LC=EC$, $C(x_0,x_0,E)=I$. Since $LBC=BLC=EBC$, $BC=C\tilde{B}(x_0,E)$ for some matrix $\tilde{B}(x_0,E)$. Also, if $\dot{} = \frac{\partial}{\partial x_0}$, $L\dot{C}=E\dot{C}$, so $\dot{C}=C\tilde{L}(x_0,E)$. Compatibility gives

$$\dot{\tilde{B}}(x_0,E) = [\tilde{B}(x_0,E), \tilde{L}(x_0,E)].$$

This is a "Kac-Moody" equation. The Riemann surface enters because the characteristic equation of $\tilde{B}$,

$$\det(\tilde{B}(x_0,E) - zI) = 0,$$

which defines an algebraic curve, is independent of x_0. Conversely, given $\dot{\tilde{B}} = [\tilde{B}, \tilde{L}]$, one can reconstruct $[L, B] = 0$.

(For example, finite-gap Hill theory is, in this way, converted to an equation in a Kac-Moody extension of $SL(2, \mathbb{R})$. On the other hand, the Toda equations can be transformed from their Kac-Moody form $\dot{\tilde{B}} = [\tilde{B}, \tilde{L}]$,

$$\tilde{B} = \begin{pmatrix} b_1 & a_1 & & 0 \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{N-1} \\ 0 & & a_{N-1} & b_N \end{pmatrix} + E\begin{pmatrix} & 0 \\ a_N & \end{pmatrix} + E^{-1}\begin{pmatrix} & a_N \\ 0 & \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} 0 & a_1 & & 0 \\ -a_1 & \ddots & \ddots & \\ & \ddots & \ddots & \\ 0 & & & 0 \end{pmatrix} + E\begin{pmatrix} & 0 \\ -a_N & \end{pmatrix} +$$

$$+ E^{-1}\begin{pmatrix} & a_N \\ 0 & \end{pmatrix},$$

into $[L, B] = 0$ — integrodifferential, like sine-Gordon, because of E^{-1}.(+) Neither of these topics has so far been studied; the Toda question is probably related to Mikhailov's Toda - sine Gordon chain).

In any case, the Adler - van Moerbecke equation (*) can easily be reconverted into $[L, B] = 0$, where $L = -\varepsilon^{-1}D - \varepsilon^{-1}(x \otimes y - y \otimes x)$, $B = \varepsilon^{-1}D^2 + \varepsilon^{-1}(x \otimes y - y \otimes x)D + \varepsilon^{-1}(x \otimes y - y \otimes x)' - x \otimes x$. These matrix operators, in turn, ought to be constructed directly from the Riemann surface $\mathcal{R}$, Baker-Ahiezer function ψ, and trace-formula function h (the one with poles at $P_0, \ldots, P_N$, $E(P_i) = E_i$)

(+) In the usual Floquet theory approach, E is just the Floquet multiplier for the discrete Schrödinger equation.

of the original Hill equation. It is not hard to see that the vector Baker-Ahiezer function Ψ defining L, B will satisfy

$$L\Psi = h\Psi,$$
$$B\Psi = h^2 E\,\Psi,$$

where h is the h from the trace formula. Ψ_i is required to go like $(\delta_{ij} + O(h^{-1}))\,e^{-E_j ht}$ at the pole P_j of h. Furthermore, $\tilde{\Psi} := \Psi e^{Eht}$ is regular at the P_j, and goes like e^{kt} at $E = \infty$. To construct L, B, therefore, one needs to find functions $\tilde{\Psi}_i$, $\tilde{\Psi}_i \sim \delta_{ij}$ at P_j and $\sim e^{kt}$ at $E = \infty$. They should be expressed in terms of the Hill $\psi(t, P)$, which goes like e^{kt} at ∞.

I have some candidates for $\tilde{\Psi}_i$, but they are so complicated that almost nothing can be checked, and I don't like them very much. This, as of the moment, is the remaining problem.

6. Conclusion.

I have tried to outline a (still very shaky) framework for thinking systematically about very recent discoveries in soliton theory. Let me once more list the main points, since very likely they were obscured by my speculations.

(1) From every linear eigenvalue problem, one can eliminate the potential (s) in favor of squared eigenfunctions. The resulting nonlinear system is itself integrable[†]. All the isospectral flows translate neatly into Hamiltonian systems for the eigenfunctions (I did not describe this).

(Recent papers have convinced me that this may be the "natural" way to represent isospectral flows; one need only compare monstrosities like the 7^{th}-order KdV with the corresponding Neumann system. I also believe that these representations may be particularly "natural" for Painlevé-type equations. We will report some results on such equations, and on some simple isospectral problems, soon; there are many other examples yet to be worked out).

(2) One can construct an L,B representation for the nonlinear system of (1), and hence establish integrability.

[†] As put by Trubowitz: 1-dimensional inverse scattering can be done because it is an integrable Hamiltonian system.

(This is a conjecture, which I tried to support above).

(3) These linear problems admit geometric interpretations.

(Like geodesic flow on ellipsoids. More precisely, I think that in each "class" of deformation problems, like the "$-D^2+q$ class", or the "D^3+qD+p class", there will be one admitting a geometric explanation. The others will be less directly geometrical, being constructed from commuting Hamiltonians).

Further down the line, one expects to encounter Lie algebras, geometry of quadrics, etc. All these matters seem to be very dimly understood; they certainly are dimly understood by me. It would help to have the many different approaches set in this language: Hirota's method, monodromy-preserving deformations, τ-functions, ... There may be much more experimentation ahead before any "real meaning" becomes clear.